Finding Satisfying Assignments by Random Walk

Rolf Wanka, Erlangen

Friedrich-Alexander-Universität
Erlangen-Nürnberg

Lehrstuhl für Informatik 12
Hardware-Software-Co-Design

Overview

- Preliminaries
- A Randomized Polynomial-time Algorithm for 2-SAT
- A Randomized $O^*(2^n)$-time Algorithm for 3-SAT
- A Randomized $O^*((4/3)^n)$-time Algorithm for 3-SAT

Preliminaries (I)

Satifiability problem SAT: Given a Boolean formula Φ in Conjunctive Normal Form (CNF) over n variables $x_1, \ldots, x_n$ and m clauses.

CNF = Conjunction of clauses;
Clause = Disjunction of literals;
Literal = variable or negation of variable

Question: Is there a truth assignment to the variables such that Φ evalutes to TRUE ?

Example for $n = 4$ and $m = 5$:

$$\Phi = (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_3) \wedge (x_1 \vee x_2) \wedge (x_4 \vee \bar{x}_3) \wedge (x_4 \vee \bar{x}_1)$$

Satisfied by

$$x_1 := \text{TRUE};\ x_2 := \text{TRUE};\ x_3 := \text{FALSE};\ x_4 := \text{TRUE}$$

Preliminaries (II)

$k \in \mathbb{N}$: For k-SAT, Φ is restricted to that each clause has exaclty k literals.

So,

$$\Phi = (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_3) \wedge (x_1 \vee x_2) \wedge (x_4 \vee \bar{x}_3) \wedge (x_4 \vee \bar{x}_1)$$

is an instance of 2-SAT.

Time complexity:

SAT is NP-complete.

3-SAT is NP-complete

2-SAT is in P.

A Randomized Polynomial-time Algorithm for 2-SAT (I)

2-SAT Algorithm ($c \in \mathbb{N}$ being an arbitrary constant):

- Start with an arbitrary truth assignment;
- Repeat up to $2cn^2$ times, terminating if all clauses are satified the following iteration:
  - Choose an arbitrary clause C that is not satisfied;
  - Choose uniformly at random one of the literals in C and switch the value of its variable;
- If a valid truth assignment has been found, return YES
- Otherwise, return NO.

Theorem: Φ is satifiable $\Rightarrow$ Pr(algo. returns YES)$\geq 1 - \frac{1}{2^c}$

A Randomized Polynomial-time Algorithm for 2-SAT (II)

Let S represent a satisfying assignment.
A_i: the truth assignment after the ith iteration.
X_i: number of variables in A_i with identical value in S

Algorithm terminates with YES if $X_i = n$.

We have

$$\begin{aligned} \Pr(X_{i+1} = 1 \mid X_i = 0) &= 1 \\ \Pr(X_{i+1} = j+1 \mid X_i = j) &\geq \frac{1}{2} \\ \Pr(X_{i+1} = j-1 \mid X_i = j) &\leq \frac{1}{2} \end{aligned}$$

A Randomized Polynomial-time Algorithm for 2-SAT (III)

Let S represent a satisfying assignment.
A_i: the truth assignment after the ith iteration.
X_i: number of variables in A_i with identical value in S

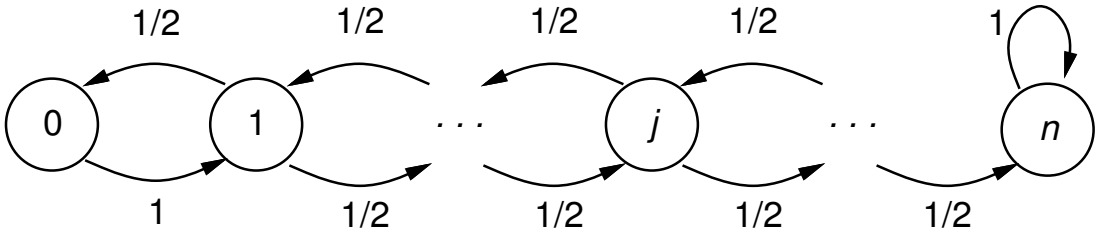
Algorithm terminates with YES if $X_i = n$.

We have

$$\begin{aligned} \Pr(X_{i+1} = 1 \mid X_i = 0) &= 1 \\ \Pr(X_{i+1} = j+1 \mid X_i = j) &= \frac{1}{2} \\ \Pr(X_{i+1} = j-1 \mid X_i = j) &= \frac{1}{2} \end{aligned}$$

A Randomized Polynomial-time Algorithm for 2-SAT (IV)

Graphical representation



h_j = expected no. of steps to reach n when starting from j

We have the system of equations:

$$\begin{aligned} h_n &= 0 \\ h_j &= \tfrac{1}{2} \cdot (h_{j-1} + h_{j+1}) + 1 \qquad \text{for } j \in \{1, \ldots, n-1\} \\ h_0 &= h_1 + 1 \end{aligned}$$

Its unique solution: $h_j = n^2 - j^2$

A Randomized Polynomial-time Algorithm for 2-SAT (V)

That means (if Φ is satisfiable, S the only valid assignment):

The expected number of iterations until the algorithm returns YES is at most n^2.

The algorithm executes $2cn^2$ iterations.
Divide the iterations into c segments $\Sigma_1, \ldots, \Sigma_c$ of $2n^2$ iterations each.
Let Z_i be the number of iterations from the start of Σ_i until S is found.
Then by Markov's inequality,

$$\Pr(Z_i \geq 2n^2) \leq \frac{E[Z_i]}{2n^2} \leq \frac{n^2}{2n^2} = \frac{1}{2}$$

$\Rightarrow \Pr(\text{algo. fails to find } S) \leq \left(\frac{1}{2}\right)^c$

A Randomized $O^*(2^n)$-time Algorithm for 3-SAT (I)

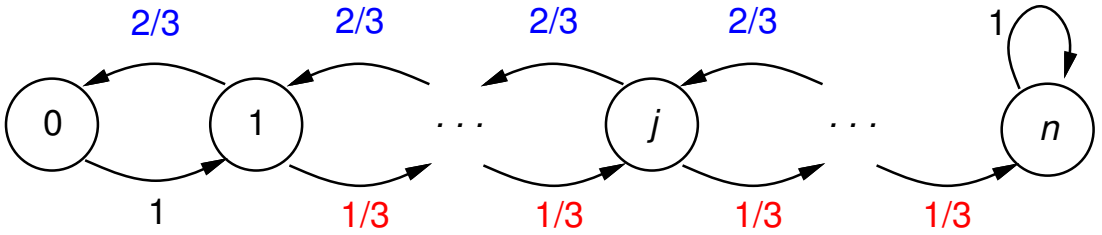
First 3-SAT Algorithm:

- Start with an arbitrary truth assignment;
- Repeat up to ℓ times, terminating if all clauses are satified the following iteration:
  - Choose an arbitrary clause C that is not satisfied;
  - Choose uniformly at random one of the literals in C and switch the value of its variable;
- If a valid truth assignment has been found, return YES
- Otherwise, return NO.

Theorem: Φ is satifiable $\Rightarrow$ The expected no. ℓ of iterations to find a valid truth assignment is $\Theta(2^n)$.

A Randomized $O^*(2^n)$-time Algorithm for 3-SAT (II)

Graphical representation assuming satisfying assignment S and counting the "correct" variables



h_j = expected no. of steps to reach n when starting from j

We have the system of equations:

$$\begin{aligned} h_n &= 0 \\ h_j &= \tfrac{2}{3} \cdot h_{j-1} + \tfrac{1}{3} \cdot h_{j+1} + 1 \qquad \text{for } j \in \{1, \ldots, n-1\} \\ h_0 &= h_1 + 1 \end{aligned}$$

Its unique solution: $h_j = 2^{n+2} - 2^{j+2} - 3(n-j)$

A Randomized $O^*(2^n)$-time Algorithm for 3-SAT (III)

Observations:

1. If A_0 is chosen u. a. r, X_0 follows a symmetric binomial distribution,

$$\Pr(X_0 = j) = \binom{n}{j} \cdot \left(\frac{1}{2}\right)^n$$

   with $E[X_0] = \frac{1}{2}n$. I. e., there is an exponentially small but non-negligible probability that A_0 matches S in significantly more than $\frac{1}{2}n$ variables.

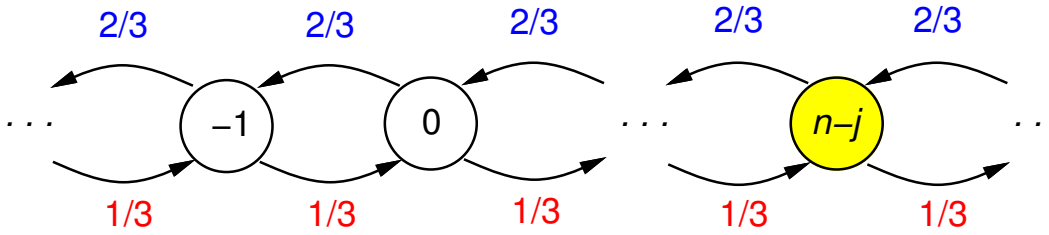
2. The algorithm is more likely to move towards 0 than towards n. The longer we run, the more likely we have moved towards 0.

A Randomized $O^*((4/3)^n)$-time Algorithm for 3-SAT (I)

Schöning's 3-SAT Algorithm:

- Repeat up to ℓ times, terminating if all clauses are satified:
  (a) Start with a truth assignment chosen u. a. r.; [Restart]
  (b) Repeat the following iterations up to n times terminating if all clauses are satified:
    (1) Choose an arbitrary clause C that is not satisfied;
    (2) Choose uniformly at random one of the literals in C and switch the value of its variable;
- If a valid truth assignment has been found, return YES
- Otherwise, return NO.

A Randomized $O^*((4/3)^n)$-time Algorithm for 3-SAT (II)

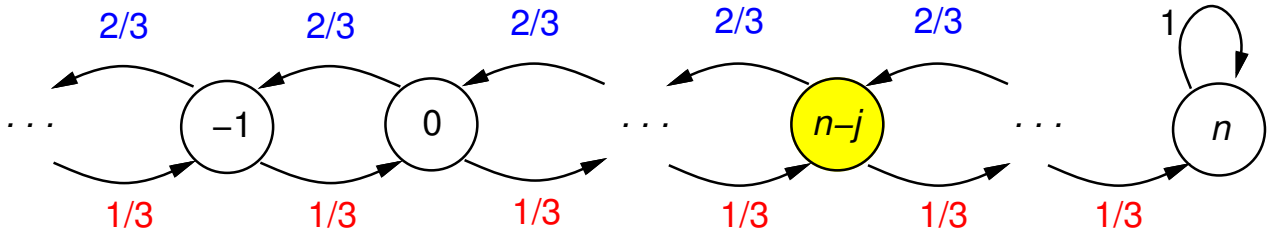


The probability of exactly k moves down and $k+j$ moves up in a sequence of $j+2k$ moves:

$$\binom{j+2k}{k} \cdot \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{j+k}$$

A Randomized $O^*((4/3)^n)$-time Algorithm for 3-SAT (III)

q_j = (lower bound on) the probability that Schöning's algorithm reaches n when it starts with an assignment with exactly j mismatches.



So,

$$q_j \geq \max_{k \in \{0,\ldots,j\}} \binom{j+2k}{k} \cdot \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{j+k}$$

In particular, with $k = j$:

$$q_j \geq \binom{3j}{j} \cdot \left(\frac{2}{3}\right)^j \left(\frac{1}{3}\right)^{2j}$$

A Randomized $O^*((4/3)^n)$-time Algorithm for 3-SAT (IV)

Let m_j denote the probability that the algorithm starts with exactly j mismatches. We have:

$$m_j = \binom{n}{j} \left(\frac{1}{2}\right)^n$$

Then, the probability of starting with exactly j mismatches and reaching n in $3j$ iterations is at least

$$m_j \cdot q_j \geq \binom{n}{j} \left(\frac{1}{2}\right)^n \cdot \binom{3j}{j} \cdot \left(\frac{2}{3}\right)^j \left(\frac{1}{3}\right)^{2j}$$

A Randomized $O^*((4/3)^n)$-time Algorithm for 3-SAT (V)

With $j = n/3$, the probability of starting with exactly $n/3$ mismatches and reaching n in n iterations is at least

$$\begin{aligned} m_{n/3} \cdot q_{n/3} &\geq \binom{n}{n/3} \left(\frac{1}{2}\right)^n \cdot \binom{n}{n/3} \cdot \left(\frac{2}{3}\right)^{n/3} \left(\frac{1}{3}\right)^{2n/3} \\ &\geq \left[\frac{1}{n+1} \cdot \left(3^{1/3} \cdot \left(\frac{3}{2}\right)^{2/3}\right)^n\right]^2 \cdot \left(\frac{1}{2} \cdot \left(\frac{2}{3}\right)^{1/3} \cdot \left(\frac{1}{3}\right)^{2/3}\right)^n \\ &= \frac{1}{(n+1)^2} \cdot \left(\frac{3}{4}\right)^n \end{aligned}$$

Hence, the expected number $E[\ell]$ of restarts until n iterations reach n is at most

$$E[\ell] \leq \frac{1}{m_{n/3} \cdot q_{n/3}} = (n+1)^2 \cdot \left(\frac{4}{3}\right)^n = O^*((4/3)^n)$$

Further Results

Schöning's algorithms can easily be generalized to k-SAT:

$$O^*\left(\left[2 \cdot \left(1 - \frac{1}{k}\right)\right]^n\right) \qquad \left(\xrightarrow{k \to \infty} O^*(2^n)\right)$$

Currently (Jan 2018), the fastest randomized algorithm for 3-SAT is due to Hertli (2011):

$$O^*(1.30704^n)$$