# Theoretical Analysis of Fair Bandwidth Sharing in Priority-based Medium Access

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In this report, we present the priority-based Access Game which describes medium access that is based on priorities for collision resolution by Game Theory. For this purpose, we make some simplifying assumptions: a) each node always has a message to send, b) each node has messages of same length, and c) no communication errors occur. Given this game, we can show that selfish behavior of nodes in this framework leads to unfair bandwidth sharing, i.e., each Equilibrium of this game is unfair to most users. We then describe a modification of this game which claims a small fraction of the available bandwidth to be unused.

It is furthermore shown, that this enhanced game has a Nash Equilibrium where bandwidth is shared fairly - although it is not a unique one. In future work, player strategies and communication protocols that enforce fair strategies have to be provided based on these observations.

# 1 Introduction

In real-time systems, access to shared resources is mainly implemented by priority-based approaches. A priority is assigned to each task, and, in the arbitration phase, the task with highest priority is granted exclusive access to the resource. In real-time systems, tasks often face hard deadlines. Due to its predictability in case of collisions, priority-based arbitration mechanisms are commonly used in distributed real-time systems, and applied for scheduling [10] as well as communication in distributed real-time networks [3].

The design of such systems is mainly done using analysis of the worst-case execution times [11]. However, when the average-case execution times of tasks significantly vary from their worst-case execution times, this approach can cause a waste of resources. For example in multimedia applications using compressed frames, time spent for video encoding/decoding may change between each frame. Generally speaking, applications may not only require hard deadlines, but also other Quality of Service (QoS) requirements with soft deadlines as well as bandwidth requirements. For task with such requirements, packet delay or even loss can be tolerated for some applications and may even be unrecognized by the user.

Another drawback is that offline designed systems are not flexible and hard to adapt online to system changes, such as removal or insertion of nodes.

The purpose of our paper is to take a closer look onto priority-based medium access from a theoretical point of view, especially on networks where Quality of Service depends on bandwidth. Here, an important issue is how to share the medium among communicating nodes. The drawback of priority-based access schemes is that selfish nodes with high priorities can block nodes with lower priorities leading to unfair bandwidth sharing, or, in the worst-case, to exclusive access of the node with highest priority.

In this paper, we will therefore use game theory to show that selfish behavior in priority-based medium access will lead to the blocking of tasks with low priorities. A major drawback is that nodes with low priorities are not able to influence the decisions of nodes with higher priorities. We present an mechanism to overcome this drawback where a small amount of the overall bandwidth has to be free. If this is not the case, all nodes receive a penalty (in our case, a utility of $0$). This gives nodes with low priorities the option to indirectly influence the decision of nodes with higher priorities.

It is furthermore shown, that this enhanced game has a Nash Equilibrium where bandwidth is shared fairly - although it is not a unique one. Nonetheless, protocols that converge to a fair strategy may be formulated based on this observation, finally leading to self-organizing communication systems.

The further outline of this paper is as follows. Section 2 gives a brief overview of game theoretic approaches to medium access done so far. We then formulate the priority-based medium access protocol by means of game theory in Section 3 and formally define the concept of fair bandwidth sharing. It is further shown, that selfish behavior leads to unfair bandwidth sharing. Section 4 presents an enhancement of the priority-based Access Game. It is shown that there exists an equilibrium with fair bandwidth sharing. Section 5 concludes this paper and gives a short overview of future work.

# 2 Overview

## 2.1 Related Work

Game theory has been often applied to study behavior of computer networks since it offers a framework to model and analyze situations in which players have to make decisions that have mutual, possibly conflicting consequences. In recent years, several game theoretical approaches to model and understand the dynamics of medium access and communication over an exclusively shared medium have been made [4]. The main focus of this research [5, 7] lies on contention-based medium access of Carrier Sense Multiple Access with Collision Detection (CSMA/CD). Such approaches use collision detection mechanisms: whenever a collision is detected, all transmissions are aborted and the senders wait for some random time, before trying to transmit data again. Especially, selfish (i.e., greedy) behavior is analyzed. The problem is that the more often nodes try to send data, the more collisions may occur and less amount of data can be transmitted in the end. The purpose of the game theoretic analysis is to find protocols for medium access to overcome this problem. Further goals are to distribute bandwidth fairly [9] or to reduce latency and meet deadlines [6]. The Equilibria of these protocols are analyzed which are commonly reached when all nodes try to maximize their profit simultaneously (known as Nash Equilibria [8]). Results show that when designing protocols properly, the aforementioned goals can be achieved.

## 2.2 The Access Game

Our approach is based on the Access Game that is introduced in [9]. It models the contention phase where multiple users content for medium access. Each user has two options: ”send” and ”wait”. The game uses mixed strategies, i.e., player $i$ sends with probability $p_i$ and waits with probability $1-p_i$. The Access Game is defined formally as follows.

**Definition 2.1** *The* Access Game $\mathcal{G}$ *is defined as tuple* $\mathcal{G} := (\mathcal{N}, (S_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$*, where* $\mathcal{N}$ *is a set of players,* $S_i$ *is the set of strategies for player* $i$*,* $S_i := \{p_i | p_i \in [0,1]\}$*, and* $u_i(\mathbf{p})$ *is the utility function of player* $i$ *depending on the strategies chosen by all players* $\mathbf{p} = (p_1, p_2, ..., p_n)$*.*

In contention based medium access, a user successfully transmits a data packet if it chooses to send and all other users choose to wait. Otherwise, collision occurs and no user is allowed to send. In the Access Game, the utility function $u_i$ is chosen to correspond to the probability that user $i$ successfully transmits a data packet.

$$u_i = Pr_i(\text{Successful}) = p_i \cdot \prod_{j\neq i}(1-p_j)$$

## 2.3 Contribution

For contention based medium access, the Access Game has a *Constrained Nash Equilibrium* leading to fair bandwidth sharing as shown in [9]. This game is adapted to the priority-based case. However, in priority-based medium access, users are able to block users with lower priorities. We will show that the standard game leads to unfair bandwidth sharing when played by *selfish* users what is due to the fact that nodes with low priorities are not able to influence the decisions of nodes with higher priorities. To overcome this drawback, we present an enhancement to the priority-based Access Game and show that there exists an equilibrium at which bandwidth is shared fairly by selfish users.

# 3 Priority-based Medium Access

Priority-based access schemes are characterized by the fact that collisions are resolved in an *arbitration phase*. In the arbitration phase, the active node with highest priority is determined. In this context, *active node* is a node that has data to send. In the following, we will use the terms *node*, *user*, *player*, and *message* interchangeably. A node specifies a controller connected to the bus. But for our further analysis, we state that each node just has one message to send and, thus, the same priority as this message. Nevertheless, this framework may be easily be mapped to the case where a node has more message types with different IDs to send.

The collision resolution is done by assigning an unique ID to each message representing its priority. In CAN, $0$ is the dominant bit, i.e., if a $1$ and a $0$ are sent simultaneously, $1$ is overwritten by $0$. In the arbitration phase, the active nodes write their IDs bit-wise onto the bus. Every node, which bit was overwritten by a dominant bit, stops arbitrating the bus further. Eventually, the node with lowest ID remains and is received by all nodes listening to the bus.

To adapt the Access Game defined in Def. 2.1 to priority-based medium access, following assumptions are made and used throughout this paper:

**A1:** Each user always has a message to send.

**A2:** Each user has messages of equal length.

**A3:** The system is stable, so no technical failures occur.

The users $i \in \mathcal{N}$ are ordered according to their priorities $prior(i)$ with

$$prior(1) < prior(2) < ... < prior(n). \tag{1}$$

At the beginning of the arbitration phase, the player with highest priority is chosen from all players that want to send and granted access to the medium. The utility function has to be adopted accordingly.

**Definition 3.1** *The* priority-based Access Game *is an Access Game defined by tuple* $\mathcal{G} = (\mathcal{N}, (S_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$ *and with utility function*

$$u_i = p_i \cdot \prod_{j>i}(1 - p_j). \tag{2}$$

Like for the classic Access Game, the utility function $u_i$ corresponds to the probability that user $i$ successfully transmits a data packet.

## 3.1 The Concept of Fairness in Access Games

However, we want to achieve fair bandwidth sharing. Although there are potentially many different ways to define fairness, we propose the following formal definition:

**Definition 3.2** *A* fair strategy $\tilde{p}$ *guarantees that each user has the same probability of getting access to the medium.*

*In that case, the following* fairness constraint *holds:*

$$u_1(\tilde{p}) = u_2(\tilde{p}) = ... = u_n(\tilde{p}) \tag{3}$$

We now want to consider a medium where $b \in [0, 1]$ determines the available bandwidth. Following Def. 3.2, a fair strategy might look as follows.

**Theorem 3.1** *A fair strategy $\tilde{p}$ of the priority-based Access Game is achieved if user $i$ sends with probability $\tilde{p}_i = \frac{b}{n-(n-i)\cdot b}$ for all $i \in \mathcal{N}$.*

**Proof 3.1** *From Def. 3.2 and (3), we have that*

$$\begin{array}{c} u_1(\tilde{p}) = u_2(\tilde{p}) = ... = u_n(\tilde{p}) \\ u_i(\tilde{p}) = \frac{b}{n}. \end{array} \quad (4)$$

*Assume that the priorities of users are given according to (1). Then for two users $i$ and $i+1$, the following relation between their sending probabilities is given when playing a fair strategy:*

$$\begin{aligned} \tilde{p}_i \cdot \prod_{j>i}(1-\tilde{p}_j) &= \tilde{p}_{i+1} \cdot \prod_{j>i+1}(1-\tilde{p}_j) \\ \tilde{p}_i \cdot (1-\tilde{p}_{i+1}) \cdot \prod_{j>i+1}(1-\tilde{p}_j) &= \tilde{p}_{i+1} \cdot \prod_{j>i+1}(1-\tilde{p}_j) \\ \tilde{p}_i \cdot (1-\tilde{p}_{i+1}) &= \tilde{p}_{i+1} \\ \tilde{p}_i &= \frac{\tilde{p}_{i+1}}{1-\tilde{p}_{i+1}}. \end{aligned} \quad (5)$$

*We now prove that $\tilde{p}_i = \frac{b}{n-(n-i)\cdot b}$ by complete induction:*

- *For $i = n$, we have $u_n(\tilde{p}) = \frac{b}{n}$ according to (4) and thus*

$$\tilde{p}_n = \frac{b}{n-(n-n)\cdot b} = \frac{b}{n}.$$

- *Under assumption that $\tilde{p}_{i+1} = \frac{b}{n-(n-(i+1))\cdot b}$ holds, we obtain using (5):*

$$\begin{aligned} \tilde{p}_i &= \frac{\tilde{p}_{i+1}}{1-\tilde{p}_{i+1}} = \frac{\frac{b}{n-(n-i-1)\cdot b}}{1-\frac{b}{n-(n-i-1)\cdot b}} = \\ &= \frac{b}{n-n\cdot b+i\cdot b} = \frac{b}{n-(n-i)\cdot b}. \end{aligned}$$

In the following section, we take a closer look on the fairness condition in the priority-based medium access when played by selfish users.

# 4 The Priority-Based Access Game played by Selfish Users

So-called *selfish users* try to maximize their own utility. When all users try to maximize their utilities simultaneously, a *Nash Equilibrium* is achieved. For further analysis, we denote the strategy vector containing the strategies of all users but user $i$ by $\mathbf{p}_{-i} = (p_1, p_2, ..., p_{i-1}, p_{i-1}, ..., p_n)$.

**Definition 4.1** *A strategy vector* $\mathbf{p}^*$ *is a* Nash Equilibrium *if*

$$u_i(p_i, \mathbf{p}^*_{-i}) \leq u_i(p^*_i, \mathbf{p}^*_{-i}) \quad \forall p_i \in S_i, \ \forall i.$$

We prove that the game results in unfair bandwidth sharing when played by selfish users.

**Theorem 4.1** *Selfish players may block players with lower priorities forever leading to unfair bandwidth sharing.*

**Proof 4.1** *Each user* $i$ *wants to maximize its utility* $u_i$*. As can be seen from (2), the best response of player* $i$ *depends on* $\prod_{j>i}(1-p_j)$*:*

1. *If* $\prod_{j>i}(1-p_j) > 0$*, then* $u_i$ *is maximized for* $p_i = 1$*. In this case, user* $i$ *blocks all users with lower priorities.*
2. *If* $\prod_{j>i}(1-p_j) = 0$*, then* $u_i$ *is maximized for any* $p_i \in [0,1]$ *and there exists at least one* $p_j = 1$ *with* $j > i$*.*

*Furthermore, the player with highest priority has utility function* $u_n = p_n$ *which is maximized for* $p_n = 1$*. Therfore,*

$$u_i(p_n, p_{-n}) = p_i \cdot \prod_{i<j<n}(1-p_j) \cdot (1-p_n) = 0 \quad \forall i < n.$$

*This means that all other users are blocked by user* $n$ *which has the highest priority.*

## 4.1 Enhancement of the priority-based Access Game

The main problem leading to unfair sharing lies in the fact that users with low priorities have no option to influence the decisions of users with higher priorities. On the other hand, users with higher priorities do not realize when overwriting users with lower priorities in the arbitration phase. The idea is to demand from the system that a small amount of the bandwidth stays free; this amount is denoted by $\epsilon$. If this is not the case, the utilities of all users are set to $0$. This gives nodes with low priorities the option to indirectly influence the decision of nodes with higher priorities, which may eventually lead to a fair solution.

The probability of the medium being free corresponds to the probability that all users decide not to send, given by

$$Pr(\text{Medium is free}) = \prod_{j \in \mathcal{N}}(1-p_j). \tag{6}$$

The claim that amount $\epsilon$ of the bandwidth stays free can be expressed as

$$Pr(\text{Medium is free}) \geq \epsilon \tag{7}$$

for our theoretical analysis. Consequently, the utility function has to be redefined:

**Definition 4.2** *The* enhanced priority-based Access Game *is a priority-based Access Game with utility function*

$$u_i = \begin{cases} p_i \cdot \prod_{j>i}(1-p_j), & \textit{if } \prod_{j\in\mathcal{N}}(1-p_j) \geq \epsilon \\ 0, & \textit{else} \end{cases} \tag{8}$$

We will now analyze the Nash Equilibria of this game and furthermore show, that following a fair strategy leads to a Nash Equilibrium.

## 4.2 Nash Equilibrium for Users

A Nash Equilibrium is achieved when all users try to maximize their payoffs. As already defined in Def. 4.1, for users being in an equilibrium no user will profit from deviating from its strategy. We now look at the impact of the enhanced game on the remaining users.

**Theorem 4.2** *Users $i \in \mathcal{N}$ are in a Nash Equilibrium when choosing strategies $\bar{\mathbf{p}}$ with*

$$\prod_{i\in\mathcal{N}}(1-\bar{p}_i) = \epsilon. \tag{9}$$

**Proof 4.2** *To prove that $\bar{\mathbf{p}}$ is a Nash Equilibrium, we show that users maximize their payoffs when choosing strategies $\bar{\mathbf{p}}$ by looking at the following two cases where the users deviate from this strategy:*

- *If $\prod_{j\in\mathcal{N}}(1-p_j) < \epsilon$, there is to much communication on the bus and, as defined in (8), the utilities are set to*

$$u_i = 0 \quad \forall i \in \mathcal{N}. \tag{10}$$

- *If $\prod_{j\in\mathcal{N}}(1-p_j) > \epsilon$, then we have following relation for strategy $p_i$ of user i:*

$$\begin{aligned} (1-p_i)\cdot\prod_{j\neq i}(1-p_j) &> \epsilon \\ (1-p_i) &> \epsilon\cdot\frac{1}{\prod_{j\neq i}(1-p_j)} \\ p_i &< 1-\epsilon\cdot\frac{1}{\prod_{j\neq i}(1-p_j)}. \end{aligned}$$

*What can be seen is that if user $i$ would choose strategy $p'_i$ defined as*

$$p'_i = 1-\epsilon\cdot\frac{1}{\prod_{j\neq i}(1-p_j)} \tag{11}$$

*its utility would be maximized [1], since*

$$u_i(p_i,\mathbf{p_{-i}}) < u_i(p'_i,\mathbf{p_{-i}}) \tag{12}$$

*and*

$$(1-p'_i)\cdot\prod_{j\neq i}(1-p_j) = \epsilon. \tag{13}$$

*Consequently, we have that $u_i(p_i,\bar{\mathbf{p}}_{-\mathbf{i}}) \leq u_i(\bar{p}_i,\bar{\mathbf{p}}_{-\mathbf{i}})$, $\forall p_i\in S_i,\ \forall i$.*

From Theorem 4.2 we know that the Nash Equilibrium has to fulfill $\prod_{j\in\mathcal{N}}(1-\bar{p}_j) = \epsilon$. We now show that there exists a Nash Equilibrium that is a fair strategy as stated in Theorem 3.1.

**Theorem 4.3** *The fair strategy $p^* = (p_1^*,...,p_n^*)$ with $p_i^* = \frac{b}{n-(n-i)\cdot b}$ is a Nash Equilibrium.*

**Proof 4.3** *The proof of this theorem is done by simply evaluating $\prod_{1\leq i\leq n}(1-p_i^*)$. Due to the nature of the extended priority-based Access Game, the available bandwidth is set to $b = 1-\epsilon$.*

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[1] Note that node $i$ may not choose $p'_i > 1-\epsilon\cdot\frac{1}{\prod_{j\neq i}(1-p_j)}$, since else the first case would hold resulting in utilities of $0$.

$$\prod_{1\leq i\leq n}(1-p_i^*) = \prod_{1\leq i\leq n}(1-\tfrac{b}{n-(n-i)\cdot b}) =$$
$$= \prod_{1\leq i\leq n}(1-\tfrac{1-\epsilon}{n-(n-i)\cdot(1-\epsilon)}) =$$
$$= \prod_{1\leq i\leq n}(1-\tfrac{1-\epsilon}{n-n+i+n\cdot\epsilon-i\cdot\epsilon}) = \prod_{1\leq i\leq n}\tfrac{i+n\cdot\epsilon-i\cdot\epsilon-1+\epsilon}{i+n\cdot\epsilon-i\cdot\epsilon} =$$
$$= \prod_{1\leq i\leq n}\tfrac{n\cdot\epsilon+(1-\epsilon)\cdot i-(1-\epsilon)}{n\cdot\epsilon+(1-\epsilon)\cdot i} = \prod_{1\leq i\leq n}\tfrac{n\cdot\epsilon+(1-\epsilon)\cdot(i-1)}{n\cdot\epsilon+(1-\epsilon)\cdot i} =$$
$$= \frac{(n\cdot\epsilon+(1-\epsilon)\cdot(1-1))\cdot\prod_{1<i\leq n}(n\cdot\epsilon+(1-\epsilon)\cdot(i-1))}{(n\cdot\epsilon+(1-\epsilon)\cdot n)\cdot\prod_{1\leq i<n}(n\cdot\epsilon+(1-\epsilon)\cdot i)} =$$
$$= \frac{n\cdot\epsilon\cdot\prod_{1\leq i<n}(n\cdot\epsilon+(1-\epsilon)\cdot i)}{n\cdot\prod_{1\leq i<n}(n\cdot\epsilon+(1-\epsilon)\cdot i)} = \epsilon \qquad (14)$$

*So, strategy $p^*$ fulfills (9) and is a Nash Equilibrium.*

# 5 Conclusion and Future Work

In this paper, we have defined the *priority-based Access Game* and shown that selfish allocation leads to unfair bandwidth sharing. Furthermore, we have introduced the *extended priority-based Access Game*, where a small fraction of the overall bandwidth has to be free. It was shown that for this game a Nash Equilibrium exists where bandwidth is shared fair, although it is no unique NE.

The results will be used in future work to analyze distributed learning schemes to provide self-organizing properties. *Multi-agent reinforcement learning* techniques [1, 2] could be used for the communication nodes to adapt their sending probabilities until an equilibrium is reached. Experiments would have to show how fast the strategies converge, i.e., how fast the system can adapt to changes, and if the equilibrium that is reached by these methods really fulfills the fairness condition.

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